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Compatibility in D-Posets

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In this paper the Boolean D-poset is defined and it is showed that every subset of a Boolean D-poset is a compatible set.

1. INTRODUCTION

The basic axiomatic models of quantum mechanics are the quantum logics $\mathcal{L}$ (Busch *et al.*, 1991) or orthoalgebras $\mathcal{A}$ (Randall and Foulis, 1981; Foulis *et al.*, 1992). Very important in this theory is the notion of a compatible subset of $\mathcal{L}$ (or $\mathcal{A}$, respectively), which represents simultaneously verifiable events.

There exist alternative models of quantum mechanics, for example, F-quantum spaces (Riečan, 1988), F-quantum posets (Dvurečenskij and Chovanec, 1988), and their generalization—the quasiorthocomplemented posets (Chovanec, 1993), where the compatibility of subsets has been studied.

The compatibility of a subset of elements in these cases means that they belong to the same Boolean subalgebra which is contained in a corresponding structure, which is the case of classical mechanics.

Recently there has appeared a new axiomatic model, D-posets, introduced in Kôpka and Chovanec (1994), which generalizes quantum logics, orthoalgebras, as well as the set of all effects (Dvurečenskij, n.d.). In this model, a difference operation is a primary notion from which it is possible to derive other usual notions that are important for measurements.

D-posets have been inspired by the possibility to introduce fuzzy set ideas into quantum structure models (Kôpka, 1992). On these structures, so-called D-posets of fuzzy sets, compatibility has been studied (Kôpka, n.d.-a).

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The aim of the present paper is to show that every subset of a so-called Boolean D-poset is a compatible set. Although the definition of a compatible subset of a D-poset is presented in such a way that for a D-poset which at the same time is a quantum logic, this notion is equivalent to the notion of compatibility in a quantum logic, we cannot say anything similar about the existence of such a Boolean subalgebra as in the case of a quantum logic. This fact calls for a new look at the compatibility in D-posets.

2. D-POSETS

Let $(P, \leq)$ be a nonempty partially ordered set (poset). A partial binary operation $\backslash$ is called a *difference* on P, and an element $b\backslash a$ is defined in P if and only if $a \leq b$, and the following conditions are satisfied:

(D1) $b\backslash a \leq b$.
(D2) $b\backslash(b\backslash a) = a$.
(D3) If $a \leq b \leq c$, then $c\backslash b \leq c\backslash a$ and $(c\backslash a)\backslash(c\backslash b) = b\backslash a$.

Let $(P, \leq, \backslash)$ be a poset with a difference and let 1 be the greatest element in P. The structure $(P, \leq, \backslash, 1)$ is called a *D-poset.*

Example 1. Let F be a family of all real functions from a nonempty set X into the unit interval $[0, 1]$. Let $\leq$ be a partial ordering on F such that $f \leq g$ if and only if $f(t) \leq g(t)$ for every $t \in X$. Let Φ: $[0, 1] \to [0, \infty)$ be an injective increasing continuous function such that $\Phi(0) = 0$. A partial binary operation $\backslash$ defined by the formula

$$(g\backslash f)(t) = \Phi^{-1}(\Phi(g(t)) - \Phi(f(t)))$$

for every $f, g \in F, f \leq g, t \in X$, is the difference on F. The system $(F; \leq, \backslash, 1(t) = 1)$ is a D-poset.

Example 2. Let $(L, \leq, \perp, 1, 0)$ be an orthomodular poset (see, e.g., Pták and Pulmannová, 1991). We put $b\backslash a = b \wedge a^{\perp}$ for every $a, b \in L, a \leq b$. Then L is a D-poset.

Let P be a D-poset. We put $a^{\perp} := 1\backslash a$ for any $a \in P$. We say that two elements a and b of P are *orthogonal*, and write $a \perp b$, if $a \leq b^{\perp}$ (or equivalently $b \leq a^{\perp}$).

The properties of a D-poset (Kôpka and Chovanec, 1994) enable us to define a sum operation on P, that is, a partial binary operation $\oplus$ on P (Dvurečenskij, n.d.; Hedlíková and Pulmannová, n.d.) given by: $a \oplus b$ is defined if and only if a and b are orthogonal and

$$a \oplus b := 1\backslash((1\backslash a)\backslash b) = 1\backslash((1\backslash b)\backslash a)$$

The partial binary operation $\oplus$ on P is commutative and associative (Hedlíková and Pulmannová, n.d.; Dvurečenskij, n.d.).

Let $F = \{a_1, \ldots, a_n\}$ be a finite sequence of P. According to Dvurečenskij (n.d.), recursively we define for $n \geq 3$

$$a_1 \oplus \cdots \oplus a_n := (a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n$$

supposing that $a_1 \oplus \cdots \oplus a_{n-1}$ and $(a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n$ exist in P. Definitionally, we put $a_1 \oplus \cdots \oplus a_n := a_1$ if $n = 1$, and $a_1 \oplus \cdots \oplus a_n := 0$ if $n = 0$. Then for any permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$ and any k with $1 \leq k \leq n$ we have

$$a_1 \oplus \cdots \oplus a_n = a_{i_1} \oplus \cdots \oplus a_{i_n},$$

$$a_1 \oplus \cdots \oplus a_n = (a_1 \oplus \cdots \oplus a_k) \oplus (a_{k+1} \oplus \cdots \oplus a_n)$$

Let P be a D-poset. We say that a finite system $F = \{a_1, \ldots, a_n\}$ of P is $\oplus$-orthogonal iff $a_1 \oplus \cdots \oplus a_n$ exists in P and write

$$a_1 \oplus \cdots \oplus a_n = \bigoplus_{i=1}^{n} a_i$$

An arbitrary system G of P is $\oplus$-orthogonal if every finite subsystem F of G is $\oplus$-orthogonal.

Definition 1. Let P be a D-poset. We say that the finite subset $F = \{a_1, \ldots, a_n\} \subseteq P$ is compatible (in P) if there exists a $\oplus$-orthogonal system G of elements of P, $G = \{g_t, t \in T\}$, such that $a_i = \oplus\{g_t; t \in T_i\}$, where T_i is the finite subset of T, for every $i = 1, \ldots, n$.

An arbitrary subset $E \subseteq P$ is compatible (in P) if every finite subset of E is compatible (in P).

3. BOOLEAN D-POSETS

In the present section we give the sufficient condition for the compatibility of a subset of a D-poset.

Let $(P, \leq)$ be a poset with the smallest element 0. Let $\ominus$ be a binary operation on P such the following conditions are satisfied for every $a, b, c \in P$:

(BD1) $a \ominus 0 = a$.
(BD2) If $a \leq b$, then $c \ominus b \leq c \ominus a$.
(BD3) $(c \ominus a) \ominus b = (c \ominus b) \ominus a$.
(BD4) $b \ominus (b \ominus a) = a \ominus (a \ominus b)$.

Proposition 1. (Kôpka, n.d.-b). Let $(P, \leq)$ be a poset with the smallest element 0 and let $\ominus$ be a binary operation on P satisfying the conditions (BD1)–(BD4). Then the following assertions are true for every $a, b, c, d \in P$:

(i) $b \ominus a \leq b$.
(ii) $a \ominus a = 0$.
(iii) If $a \leq b$, then $a \ominus b = 0$.
(iv) $(c \ominus a) \ominus (c \ominus b) = (b \ominus a) \ominus (b \ominus c)$.
(v) If $a \leq b \leq c$, then $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.
(vi) If $a \leq b$, then $b \ominus (b \ominus a) = a$.
(vii) If $b \leq c$, then $b \ominus a \leq c \ominus a$.
(viii) If $b \leq c$, then $(c \ominus a) \ominus (b \ominus a) = (c \ominus b) \ominus ((a \ominus b) \ominus (a \ominus c))$.
(ix) If $b \ominus a = 0$ and $a \ominus b = 0$, then $a = b$.
(x) If $a, b \leq c$ and $c \ominus a = b \ominus a$, then $c \ominus b = a \ominus b$.
(xi) Let 1 be the greatest element in P, $a, b, c \in P$. If $a \leq c$, $a \leq b$, and $c \ominus a = b \ominus a$, then $b = c$.

Proposition 1 proves that the binary operation $\ominus$ satisfies the conditions (D1)–(D3), i.e., it is a difference on P.

Definition 2. Let $(P, \leq)$ be a poset with the smallest element 0 and with the greatest element 1. Let $\ominus$ be a binary operation on P satisfying the conditions (BD1)–(BD4). The system $(P, \leq, 0, 1, \ominus)$ is called a *Boolean D-poset.*

Example 3. Let the binary operation $\ominus$ on the family of all real functions F from Example 1 be defined by the following formula:

$$(g \ominus f)(t) = \begin{cases} \Phi^{-1}(\Phi(g(t)) - \Phi(f(t))) & \text{if } f(t) \leq g(t) \\ 0 & \text{if } f(t) > g(t) \end{cases}$$

Then $(F, \leq, \ominus, 1, 0)$ be a Boolean D-poset.

Example 4. Every MV-algebra (Chang, 1959) is a Boolean D-poset.

We remark that every Boolean D-poset P is a D-poset and the binary operation $\ominus$ on P generates the binary operation $\dotplus$ on P defined via $a \dotplus b := (a^{\perp} \ominus b)^{\perp}$, where $a^{\perp} = 1 \ominus a$ for every $a \in P$. The operation $\dotplus$ on P has the following properties:

Proposition 2 (Kôpka, n.d.-b). Let $(P, \leq, 0, 1, \ominus)$ be a Boolean D-poset. Then the following assertions are true for every $a, b, c, \in P$:

(1) $a \dotplus b \geq a, b$.
(2) $a \dotplus b = b \dotplus a$ (commutativity).

(3) $(a \dot{+} b) \dot{+} c = a \dot{+} (b \dot{+} c)$ (associativity).
(4) $a \dot{+} 0 = a$.
(5) If $a \leq b$, then $a \oplus c \leq b \oplus c$.
(6) $a \dot{+} (b \ominus a) = b \dot{+} (a \ominus b)$.
(7) $(b \ominus a) \dot{+} (b \ominus (b \ominus a)) = b$.
(8) If $a \leq b$, then $a \dot{+} (b \ominus a) = b$.
(9) If $a \leq c$, $b \leq c \ominus a$, then $a \dot{+} b = c \ominus ((c \ominus a) \ominus b)$.

Remark 1. Let P be a Boolean D-poset, let G be a system of elements of P, $G = \{g_t, t \in T\}$. Then the system G is $\dot{+}$-orthogonal (in the sense of D-posets) if the sums $\dot{+}\{g_t, t \in T_1\}$, $\dot{+}\{g_t, t \in T_2\}$ are orthogonal, i.e., $\dot{+}\{g_t, t \in T_1\} \leq 1 \ominus (\dot{+}\{g_t, t \in T_2\})$, for finite subsets T_1 and T_2 of T, such that $T_1 \cap T_2 = \emptyset$.

Theorem 1. Let $(P, \leq, 0, 1, \ominus)$ be a Boolean D-poset. Then an arbitrary subset E of P is a compatible set (in P).

Proof. It suffices to prove that for every finite subset E of P, $E = \{a_1, \ldots, a_n\}$, there exists a $\dot{+}$-orthogonal system G of elements of P, $G = \{g_t; t \in T\}$, such that $a_i = \dot{+}\{g_t; t \in T_i\}$, where T_i is a finite subset of T, $i = 1, \ldots, n$. The existence of the system G will be proved by mathematical induction according to the number of the elements of the set E.

1. Let $n = 2$, i.e., $E = \{a, b\}$. Then the system

$$G = \{a \ominus b, b \ominus a, a \ominus (a \ominus b) = b \ominus (b \ominus a)\}$$

is $\dot{+}$-orthogonal and $a = (a \ominus (a \ominus b)) \dot{+} (a \ominus b)$, $b = (b \ominus (b \ominus a)) \dot{+} (b \ominus a)$.

2. We assume that the previous assertion holds for every subset E of P containing $n - 1$ elements, i.e., if $E = \{a_1, \ldots, a_{n-1}\}$ then there exists a $\dot{+}$-orthogonal system G of elements of P, $G = \{g_t, t \in T\}$, such that $a_i = \dot{+}\{g_t, t \in T_i\}$, where T_i is a finite subset of T, $i = 1, \ldots, n - 1$.

Without loss of generality we assume that

$$G = \left\{g_t, t \in \bigcup_{i=1}^{n-1} T_i\right\} = \{g_1, \ldots, g_k\}$$

Let now $E = \{a_1, \ldots, a_{n-1}, a\}$. We put

$$b_0 = a$$

$$b_i = b_{i-1} \ominus g_i \qquad \text{for every} \quad i = 1, \ldots, k$$

It is evident that $b_{i-1} \geq b_i$ for every $i = 1, \ldots, k$.

Now we construct the system of elements of P in the following way:

$$c_i = b_{i-1} \ominus b_i \qquad \text{for every} \quad i = 1, \dots, k$$
$$c_{k+1} = b_k$$

By the properties of the binary operation $\ominus$ we have

$$c_1 = b_0 \ominus b_1 = a \ominus (a \ominus g_1) = g_1 \ominus (g_1 \ominus a) \leq g_1$$
$$c_2 = b_1 \ominus b_2 = (a \ominus g_1) \ominus ((a \ominus g_1) \ominus g_2)$$
$$= g_2 \ominus (g_2 \ominus (a \ominus g_1)) \leq g_2$$
$$\vdots$$
$$c_k = b_{k-1} \ominus b_k = ((a \ominus g_1) \ominus \cdots \ominus g_{k-1})$$
$$\ominus (((a \ominus g_1) \ominus \cdots \ominus g_{k-1}) \ominus g_k)$$
$$= g_k \ominus (g_k \ominus ((a \ominus g_1) \ominus \cdots \ominus g_{k-1})) \leq g_k$$
$$c_{k+1} = b_k = (a \ominus g_1) \ominus \cdots \ominus g_k$$

Then the system $\{g_1 \ominus c_1, \dots, g_k \ominus c_k, c_1, \dots, c_k, c_{k+1}\}$ is $\dotplus$-orthogonal, $g_i = (g_i \ominus c_i) \dotplus c_i$, for every $i = 1, \dots, k$, and

$$c_1 \dotplus \cdots \dotplus c_{k+1}$$
$$= (c_1 \dotplus \cdots \dotplus c_{k-1}) \dotplus (c_k \dotplus c_{k+1})$$
$$= (c_1 \dotplus \cdots \dotplus c_{k-1}) \dotplus b_{k-1} = (c_1 \dotplus \cdots \dotplus c_{k-2}) \dotplus (c_{k-1} \dotplus b_{k-1})$$
$$= (c_1 \dotplus \cdots \dotplus c_{k-2}) \dotplus ((b_{k-2} \ominus b_{k-1}) \dotplus b_{k-1})$$
$$= (c_1 \dotplus \cdots \dotplus c_{k-2}) \dotplus b_{k-2}$$
$$\cdots = (a \ominus b_1) \dotplus b_1 = a \quad \blacksquare$$

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